On the Mathematical Symmetry in Physical Cause-Effect Equations

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I. EXPRESSION OF CAUSALITY

There is no mathematical mean to distinguish cause terms (RHS) and effect terms (LHS) in the current mathematics. However, from a physics point of view, RHS and LHS are not inter-exchangeable because it means "time arrow" or cause-effect relationship. For example,

$$\frac{d\vec{p}}{dt} = \vec{F} \tag{1}$$

means change of momentum caused by impeding force, but

$$\vec{F} = \frac{d\vec{p}}{dt}, \tag{2}$$

should mean a force caused by the change of momentum, which is physically incorrect because the correct expression of the inertial force is:

$$\vec{F} = -\frac{d\vec{p}}{dt}. \tag{3}$$

Just changing LHS and RHS in physics equation with time derivative should require an addition of a minus sign to make the equation physically meaningful.

In order to avoid confusion, we introduce a new asymmetric equal sign "$\Leftarrow$" which expresses the causality relations of Newton's Law, Faraday's Law, and Ampere's Law.

$$\frac{d\vec{p}}{dt} \Leftarrow \vec{F}, \tag{4}$$

$$\mathrm{rot}\vec{E} \Leftarrow -\frac{\partial \vec{B}}{\partial t}, \tag{5}$$

$$\mathrm{rot}\vec{H} \Leftarrow \frac{\partial \vec{D}}{\partial t} + \vec{J}. \tag{6}$$

with Poisson's equation and monopole prohivition

$$\begin{aligned} \mathrm{div}\cdot\vec{D} &= \rho_c \\ \mathrm{div}\cdot\vec{B} &= 0 \end{aligned} \tag{7}$$

Obvious restrictions related to this new sign "$\Leftarrow$" are:

1. One may not freely exchange or move terms between RHS and LHS of the $\Leftarrow$ sign.

2. Therefore, one may not freely add/subtract the cause terms on the LHS of the $\Leftarrow$ sign (or one may not freely add/subtract the effect terms on the RHS of the $\Leftarrow$ sign).

3. Therefore, elimination of terms must be done in such a way that the effect terms is substituted to the cause terms.

They directly lead to energy equations

$$\frac{d}{dt}\left(\frac{p^2}{2m}\right) \angle\!\!= \ \frac{\vec{p}\cdot\vec{F}}{m}, \tag{8}$$

$$\begin{aligned} \mathrm{div}\cdot\vec{S} &= -\vec{E}\cdot\mathrm{rot}\vec{H} + \vec{H}\cdot\mathrm{rot}\vec{E} \\ &\angle\!\!= \ -\vec{J}\cdot\vec{E} - \frac{\partial}{\partial t}U_{em}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \vec{S} &= \ \vec{E}\times\vec{H} \\ U_{em} &= \ \vec{E}\cdot\vec{D}/2 + \vec{B}\cdot\vec{H}/2 \end{aligned}$$

The wave form of Maxwell's equations in space is

$$-\nabla^2\vec{E} \angle\!\!= \ -\mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\vec{E} - \mu_0\frac{\partial}{\partial t}\vec{J}, \tag{10}$$

with Poissons equation (7):

$$\begin{aligned} \mathrm{div}\cdot\vec{E} &= \frac{\rho_c}{\epsilon_0}, \\ \mathrm{div}\cdot\vec{B} &= 0, \end{aligned}$$

and momentum equation that relates $\vec{J}$ and $\vec{E}$ in the form of $\vec{J} \angle\!\!= \ f(\vec{E})$. The equation (10) means that Electric field is determined by (a) propagating electric-magnetic wave, (b) change in the electric current, and (c) accumulated space charge, but not opposite.

II. FLUID EQUATION

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\vec{u}) = 0, \tag{11}$$

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \nabla\cdot(\rho\mathbf{uu}) \Leftarrow \rho\vec{K} - \nabla\cdot\mathbf{P},$$

or

$$\rho\frac{\partial}{\partial t}\vec{u} + \rho(\vec{u}\cdot\nabla)\vec{u} \Leftarrow -\nabla P + \rho_c\vec{E} + \vec{J}\times\vec{B} + \rho\vec{G} + \nu\Delta\vec{u} \tag{12}$$

$$\text{Function}(\rho, p, T) = 0, \tag{13}$$

where $\rho\vec{K} \simeq \vec{J}\times\vec{B}$ in ideal case.

III. MHD EQUATION

<u>MHD equations</u>: (One Fluid)

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho + (\vec{u}\cdot\nabla)\rho &= -\rho(\nabla\cdot\vec{u}) \\
\rho\frac{\partial}{\partial t}\vec{u} + \rho(\vec{u}\cdot\nabla)\vec{u} &\Leftarrow -\nabla P + \rho_c\vec{E} + \vec{J}\times\vec{B} \\
P &= P(\rho) \\
\frac{1}{\mu_0}\nabla\times\vec{B} &\Leftarrow \vec{J} \\
\nabla\times\vec{E} &\Leftarrow -\frac{\partial}{\partial t}\vec{B} \\
\eta(\tau_e\frac{\partial}{\partial t}\vec{J} + \vec{J}) &\Leftarrow \vec{E} + \vec{u}\times\vec{B} - \frac{1}{en_e}\vec{J}\times\vec{B}, \qquad (14)
\end{aligned}
$$

where the Ohm's law (14) is a causality equation because it is rooted to electron momentum equation. When η is nearly zero, very small motional electric field $\vec{E}' = \vec{E}+\vec{u}\times B$ or very small Hall term will cause large $\partial/\partial t\vec{J}$, that quickly adjust $\vec{E}'$ or Hall term becoming zero. If this process is very fast compared to plasma motion (i.e., large-scale slow motion), we can assume $\vec{E}' \sim 0$ as the result of quick feedback.

Only in such case, we can use induction equation, but in a form of

$$\nabla\times(\vec{u}\times\vec{B}) \Leftarrow -\frac{\partial}{\partial t}\vec{B} \qquad (15)$$

i.e., flow or magnetic field is modified by the change in the magnetic field, but not in the opposite ways.

IV. THERMODYNAMICS LAWS

First Law:

$$dU \;\angle\!\!= dQ' + dW', \tag{16}$$

where dQ' is the total heat given to the fluid, dW' is the total work actually given to the fluid, and dU is the total change of the internal energy. By setting $dW' = -PdV$

$$\begin{aligned}&\frac{\partial}{\partial t}[\rho U_{th}] + \nabla\cdot[\rho U_{th}\vec{u}] \\ &\angle\!\!= -\nabla\cdot\vec{\Theta} + [\rho\vec{u}\cdot\vec{K}]_{dissipate} - \sum_{ij} P_{ij}\frac{\partial u_j}{\partial x_i},\end{aligned} \tag{17}$$

where $\vec{\Theta}$ is the heat flow.

(**??**) + (17), or by setting $dU = U_{th} + u^2/2$ in (16) and including the pressure gradient force VdP in addition to PdV,

$$\begin{aligned}&\frac{\partial}{\partial t}[\rho(\frac{u^2}{2} + U_{th})] + \nabla\cdot[\rho\vec{u}(\frac{u^2}{2} + U_{th})] \\ &\angle\!\!= -\nabla\cdot\vec{\Theta} + \rho\vec{u}\cdot\vec{K} - \nabla\cdot(\mathbf{P}\cdot\vec{u}),\end{aligned} \tag{18}$$

Adding the Pointing's law, we finally have

$$\begin{aligned}&\frac{\partial}{\partial t}[\rho(\frac{u^2}{2} + U_{th})] + \nabla\cdot[\rho\vec{u}(\frac{u^2}{2} + U_{th}) + \vec{S}] \\ &\angle\!\!= -\frac{\partial}{\partial t}U_{em} - \nabla\cdot\vec{\Theta} + \rho\vec{u}\cdot\vec{g} - \nabla\cdot(\mathbf{P}\cdot\vec{u}),\end{aligned} \tag{19}$$

Second Law:

$$dS \;\angle\!\!= (\frac{dQ'}{T})_{ideal} = \frac{dU}{T} - \frac{dW''}{T} \geq 0, \tag{20}$$

where $dW'' - P\nabla\cdot\vec{u}$ is the total net work during constant pressure whereas $dW' = -\sum_{ij} P_{ij}\partial u_j/\partial x_i$ corresponds to the total work for given pressure. Then:

$$
\begin{aligned}
&\rho T[\frac{\partial}{\partial t}S + \nabla\cdot(\vec{u}S)] \\
&\eqslantless \frac{\partial}{\partial t}[\rho U_{th}] + \nabla\cdot[\rho U_{th}\vec{u}] - [\rho\vec{u}\cdot\vec{K}]_{linear} + \rho P\frac{d}{dt}(\frac{1}{\rho}), \\
&\eqslantless -\nabla\cdot\vec{\Theta} + [\rho\vec{u}\cdot\vec{K}]_{dissipate} + \rho\chi(\nabla\cdot\mathbf{u})^2 \\
&\quad + \frac{\rho\mu}{2}[\sum_{ij}(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}) - 2\nabla\cdot\mathbf{u}]^2 \\
&\geq 0, \qquad (21)
\end{aligned}
$$

where μ and χ are viscosity rates.

V. DISTRIBUTION FUNCTION

Start from

$$\frac{\partial}{\partial t} f + \frac{dx}{dt}\frac{\partial}{\partial x} f + \frac{dv}{dt}\frac{\partial}{\partial v} f \mathrel{\angle\!\!\!=} \frac{\partial_c f}{\partial t}, \tag{22}$$

or

$$\frac{\partial}{\partial t} f + \frac{dx}{dt}\frac{\partial}{\partial x} f + \frac{dv}{dt} = 0, \tag{23}$$

Integrate over the velocity space.

$$\int h \frac{\partial}{\partial t} f d\Gamma + \int h \vec{v} \cdot \nabla f d\Gamma + \int h \frac{d\vec{v}}{dt} \cdot \nabla_v f d\Gamma = 0,$$

where $h = 1$ or $h = v_i$ or $h = v^2$. Using the relation:

$$\begin{aligned} \int h(v) \frac{\partial}{\partial t} f d\Gamma &= \frac{\partial}{\partial t} < h(v) >, \\ \int h(v) \vec{v} \cdot \nabla f d\Gamma &= \nabla \cdot < h(v) \vec{v} >, \\ \int h(v) \frac{d\vec{v}}{dt} \cdot \nabla_v f d\Gamma &= - < \nabla_v \cdot (h(v) \frac{d\vec{v}}{dt}) > \\ &= - < \frac{d\vec{v}}{dt} \cdot \nabla_v h(v) >, \end{aligned}$$

where we used $d\vec{v}/dt$ is a function of $\vec{v} \times \vec{B}$ but not $\vec{v}$.

$$\frac{\partial}{\partial t} n + \nabla \cdot (n\vec{u}) = 0, \tag{24}$$

$$\begin{aligned} \frac{\partial}{\partial t}(n u_i) + \nabla \cdot (n \vec{u} u_i) &= -(\nabla \cdot \mathbf{P})_i + < \frac{dv_i}{dt} >, \\ &\mathrel{\angle\!\!\!=} -(\nabla \cdot \mathbf{P})_i + < \frac{F_i}{m} >, \end{aligned} \tag{25}$$

$$\begin{aligned} \frac{\partial}{\partial t} < (u + v_{th})^2 > + \nabla \cdot < v^2 \vec{v} > &= < 2 \frac{d\vec{v}}{dt} \cdot \vec{v} >, \\ &\mathrel{\angle\!\!\!=} 2 < \frac{\vec{F} \cdot \vec{v}}{m} >, \end{aligned} \tag{26}$$

Thus we have exactly the same equations as (11), (12), and (18)

VI. APPLICATION

Using the new algebra, one may automatically remove wrong physical interpretations when the basic equations are modified into more complicated forms. For example, in magnetohydrodynamics, the vorticity equation can be rewritten for fast mode as:

$$\rho \frac{d}{dt}\frac{(\nabla \times \mathbf{u})_{\parallel}}{B} \mathrel{\angle\!\!=} -\nabla_{\perp}\cdot\mathbf{J}_{\perp} + \mathbf{J}_{iner}\cdot\frac{\nabla\rho}{\rho} + (\nabla P + \rho\frac{d\mathbf{u}}{dt})\cdot(\frac{\mathbf{J}_{curv} + \mathbf{J}_{\nabla B}}{P}) \quad (27)$$

or

$$\frac{d}{dt}\Omega \mathrel{\angle\!\!=} \frac{\nabla_{\perp}\cdot\vec{J}_{\perp}}{\rho} \quad (28)$$

under proper assumptions where $\Omega = (\mathrm{rot}\vec{u})\cdot\vec{B}/B^2$ [3]. Since this equation is rooted in the Newton's law, the cause-effect relation is clear, i.e., one may not interpret is as "divergence of $\vec{J}_{\perp}$ caused by vorticity but not opposite.

However, there are a numbers of papers published with a statement such as "vorticity (Ω) is a cause of field-aligned current ($\nabla_{\perp}\cdot\vec{J}_{\perp}$)" [4]. The new equal sign will remove this type of confusion.

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